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Link between solitary waves and projective Riccati equations

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Abstract. Many solitary wave solutions of nonlinear partial differential equations can be written as a polynomial in two elementary functions which satisfy a projective (hence linearizable) Riccati system. From that property, we deduce a method for building these solutions by determining only a finite number of coefficients. This method is much shorter and obtains more solutions than the one which consists of summing a perturbation series built from exponential solutions of the linearized equation. We handle several examples. For the Hénon–Heiles Hamiltonian system, we obtain several exact solutions; one of them defines a new solitary wave solution for a coupled system of Boussinesq and nonlinear Schrödinger equations. For a third order dispersive equation with two monomial nonlinearities, we isolate all cases where the general solution is single valued.

1. Introduction

Solitary wave solutions of a nonlinear partial differential equation (NLPDE) $E(u) = 0$ in the unknown $u(x,t)$ are solutions of the ordinary differential equation (ODE) obtained by the reduction $u(x,t) \to u(\xi = x - ct)$.

There previously existed two main methods for finding such solutions.

The first one [1–5] represents u as the sum of a Taylor series in exponential solutions of the linearized equation. It requires solving the recurrence relation for the series coefficients and finding the sum of the Taylor series; this method is specially adapted to solitary wave solutions expressible as geometrical series of exponentials and derivatives of such series, like $\mathrm{sech}, \tanh, \mathrm{sech}^2 = \tanh', \ldots$. However, since it starts from the linearized equation, it misses by construction any solitary wave whose speed c and wavevector k are not linked by the dispersion relation, e.g.

$$u = k^2 \left[1 - \tfrac{3}{2}\tanh^2(\tfrac{1}{2}k\xi)\right] \qquad c = \tfrac{7}{2}k^4 \tag{1}$$

a solution (not vanishing at $\xi = \pm\infty$) of the higher order Korteweg–de Vries equation

$$u_t + (u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3)_x = 0 \tag{2}$$

whose dispersion relation is $c = k^4$.

The second consists of looking for u as a polynomial in a variable which satisfies either a Riccati equation [6,7] or a degenerate elliptic equation [8], or a non-degenerate elliptic equation [9]; in the case of a Riccati subequation, this method is equivalent to performing the Weiss truncation procedure [10] in the invariant formulation of Painlevé analysis [11, 12].

This second method avoids solving a recurrence relation and summing a series, operations which can sometimes be difficult. However, some physically interesting solutions escape it, e.g.

$$u = k^2 \frac{1 + 2\cosh k\xi}{(\cosh k\xi + 2)^2} \qquad c = k^4 \tag{3}$$

the solitary wave vanishing at $\xi = \pm\infty$, i.e. the one-soliton of the Kaup–Kupershmidt (KK) equation [13, 14]

$$u_t + \left(u_{xxxx} + 30uu_{xx} + \tfrac{45}{2}u_x^2 + 60u^3\right)_x = 0. \tag{4}$$

The method we present here keeps the idea of a subequation and selects this subequation so as not to miss solitary waves of type (3). It consists of representing u as a polynomial in two variables which satisfy a system of two coupled Riccati ODEs of projective type and, consequently, in determining a *finite* number of coefficients.

2. Two elementary solitary waves

These are the bell-shaped and kink-shaped waves defined by the two functions

$$\sigma(\theta) = \frac{K}{\cosh\theta + \mu} \qquad \tau(\theta) = \frac{\sinh\theta}{\cosh\theta + \mu} \tag{5}$$

with μ and K constant.

They represent the general two-parameter solution of the coupled system of projective Riccati equations [15]

$$\sigma' = -\sigma\tau \qquad \tau' = -\tau^2 - \frac{\mu}{K}\sigma + 1 \tag{6}$$

which admits the first integral

$$\left[\frac{1}{\sigma} - \frac{\mu}{K}\right]^2 - \frac{\tau^2}{\sigma^2} = K^{-2}. \tag{7}$$

Variable θ is complex, so that the two functions can be trigonometric or hyperbolic. Both σ and τ have simple movable poles (i.e. whose location depends on integration constants) in the θ complex plane, except for $\mu = \pm 1$, in which case σ has double poles. In order to prevent this change in the pole order of σ, we choose $K = \sqrt{\mu^2 - 1}$ and consider the system

$$\begin{aligned} &\sigma' = -\sigma\tau, \tau' = -\tau^2 - \mu_0\sigma + 1 \ \text{modulo}\ 1 - \tau^2 - 2\mu_0\sigma + \sigma^2 = 0 \\ &\mu_0 = \frac{\mu}{\sqrt{\mu^2 - 1}} \end{aligned} \tag{8}$$

with the restriction $\mu_0^2 \neq 1$.

Reciprocally, the general solution of (8) is (5) if $\mu_0^2 \neq 1$; otherwise it is

$$\mu_0^2 = 1: \qquad \sigma = \tfrac{1}{2}\mu_0(1 \mp \tanh \tfrac{1}{2}\theta) \qquad \tau = \tfrac{1}{2}(\pm 1 + \tanh \tfrac{1}{2}\theta). \tag{9}$$

However, the representation of $\tanh(\Theta)$ in the basis (σ, τ) as defined by the system (8) is not unique since it is either τ with $\theta = \Theta, \mu_0 = 0$ or $\tau \pm \sigma$ with $\theta = 2\Theta +$ constant and μ_0 arbitrary; indeed, the function $T(\theta) = \tau(\theta) \pm \sigma(\theta)$ satisfies the Riccati equation

$$\mu_0 \text{ arbitrary}: \qquad T(\theta) = \tau(\theta) \pm \sigma(\theta) \qquad T' = \tfrac{1}{2}(1 - T^2). \tag{10}$$

Elimination of either σ or τ between (8) provides

$$\sigma'^2 = \sigma^2(1 - 2\mu_0\sigma + \sigma^2) \qquad \sigma'' = \sigma(2\sigma^2 - 3\mu_0\sigma + 1) \tag{11}$$

which is a degenerate elliptic equation, and

$$\tau'' + 3\tau\tau' + \tau^3 - \tau = 0. \tag{12}$$

Equations (11) and (12) are linearizable by the transformation $\sigma = 1/\psi, \tau = \psi'/\psi$

$$\psi'' - \psi + \mu_0 = 0 \qquad \psi''' - \psi' = 0. \tag{13}$$

In the following, we sometimes denote for brevity σ and τ as sechm and tanhm ('modified' sech and tanh functions respectively) in the case $\mu_0(\mu_0^2 - 1) \neq 0$.

3. Method for finding solitary wave solutions

This consists of finding, if they exist, such solitary wave solutions as polynomials in the two functions σ and τ. These two functions are evaluated at a point $\theta = \theta(\xi)$, $\theta' \neq 0$. The class of equations to which the method applies is made of the NLPDE $E(u) = 0$ polynomial in u and its derivatives. This is not a strong restriction since, e.g. the sine-Gordon equation

$$u_{xt} = \sin u \tag{14}$$

is polynomial in the variable $U = e^{iu}$.

The first step consists of determining the polynomial degree P of the solution u in (σ, τ), which must be a positive integer. Again at this stage, some transformation $u \to u^\alpha$ may be in order to satisfy this requirement.

In the second step, one defines u as the most general polynomial in σ, τ with a global degree P in (σ, τ) and a degree one in τ

$$u = \sum_{l=0}^{1} \sum_{j=0}^{P-l} c_{j,l}(\xi)\sigma^j(\theta)\tau^l(\theta) \qquad (c_{P,0}, c_{P-1,1}) \neq (0,0) \tag{15}$$

then one puts the LHS $E(u)$ under the same canonical form by eliminating any derivative of (σ, τ) and any power of τ higher than one with the definition (8) of the projective Riccati system

$$E(u) = \sum_{l=0}^{1} \sum_{j=0}^{Q-l} E_{j,l} \sigma^j \tau^l . \tag{16}$$

The next steps, described below, consist of solving the set of $2Q+1$ determining equations

$$\forall j, l : \qquad E_{j,l}(\mu_0, \theta', c, \{c_{m,n}\}) = 0 \qquad (\mu_0^2 - 1)\theta' \neq 0 \tag{17}$$

for the $2P+4$ unknowns $\mu_0, \theta', c, c_{m,n}$, in a way which avoids finding several times the same solution under different representations.

The third step consists of assuming $\mu_0(\mu_0^2 - 1) \neq 0$ in the determining equations, then solving them; any solution u polynomial in the single function $\tau \pm \sigma$ must be discarded at this stage, in order to ensure a unique representation for each solution, for such a solution will be found at the next step.

The fourth step consists of setting $\mu_0 = 0$ in the determining equations (17), then solving them; one thus obtains all solutions polynomial in sech and tanh . Again, any solution u polynomial in the single function $\tau \pm \sigma$ must be discarded. Some solutions found at this fourth step may be the particular case $\mu_0 = 0$ of solutions found at the third step.

Last, among the mathematical solutions found in the above manner, one may want to only keep the ones which satisfy some physical requirements, e.g. $u \to 0$ when $\xi \to \pm\infty$.

4. Examples

4.1. Coupled Boussinesq and nonlinear Schrödinger system, Hénon–Heiles system

The coupled PDE system

$$-\delta U_{tt} + \lambda(U_{xx} + \gamma U^2)_{xx} + \kappa \left[|\Phi|^2\right]_{xx} = 0 \tag{18a}$$
$$-\mathrm{i}\Phi_t + \Phi_{xx} + U\Phi = 0 \tag{18b}$$

with $\gamma, \delta, \kappa, \lambda$ real parameters, describes the evolution of modulational instabilities in plasmas [16, 17]. When one looks for solitary waves [18] defined as

$$U = u(\xi) \qquad \Phi = \varphi(\xi) \mathrm{e}^{\mathrm{i}(Kx - \Omega t)} \qquad \xi = x - Ct \tag{19}$$

where u, φ are real functions and C, K, Ω real parameters, the first equation (18*a*) can be integrated twice; imposing the physical boundary conditions that u and φ vanish as $\xi \to \pm\infty$, one finally obtains the relation $C + 2K = 0$ and two coupled nonlinear ODEs

$$\lambda u_{\xi\xi} + (1 - 4\delta K^2)u + \lambda\gamma u^2 + \kappa\varphi^2 = 0 \tag{20a}$$
$$\varphi_{\xi\xi} - (\Omega + K^2)\varphi + u\varphi = 0. \tag{20b}$$

These ODEs, which also come out of the same solitary wave reduction for a coupled system of Korteweg–de Vries and nonlinear Schrödinger equations [19,20], are very well known in Hamiltonian chaos. They indeed identify to the Hamilton equations for the Hénon–Heiles (HH) system [21]

$$H \equiv \tfrac{1}{2}(q_{1,\xi}^2 + \varepsilon q_{2,\xi}^2 + c_1 q_1^2 + c_2 \varepsilon q_2^2) + \alpha\varepsilon q_1 q_2^2 - \tfrac{1}{3}\beta q_1^3 = E \qquad (21a)$$

$$q_{1,\xi\xi} + c_1 q_1 - \beta q_1^2 + \varepsilon\alpha q_2^2 = 0 \qquad (21b)$$

$$q_{2,\xi\xi} + c_2 q_2 + 2\alpha q_1 q_2 = 0 \qquad (21c)$$

in which $\alpha, \beta, c_1, c_2, \varepsilon$ are real parameters; these equations are invariant under $q_2 \to -q_2$, and the parameter ε, taken as unity in the HH system, accounts for the sign of $\kappa\lambda$ in the physical equations.

In order to apply the method described in section 3, one could express that q_1 and q_2, assumed polynomial in (σ, τ), satisfy the system (21*b*) and (22*c*). For three reasons, this is not what should be done.

(a) The invariance by parity on q_2 implies that the system written for (q_1, q_2^2) is also polynomial; then, the assumption q_2 polynomial in (σ, τ) could miss solutions where the variable q_2^2 evaluates to a polynomial which is not the square of a polynomial in (σ, τ).

(b) For such a simple system, where elimination of either field is immediate, only one polynomial assumption should be made, either on q_1 or on q_2^2, again to avoid missing some solutions not polynomial in both variables; note that the assumption q_1 polynomial implies q_2^2 polynomial, but not conversely.

(c) The joint assumption (q_1, q_2) polynomial leads to too many subcases, where the polynomial degrees are $(2,2), (2,1), (2,0)$ as will be seen below, while the consideration of q_1 only leads to a single case of polynomial degree.

The elimination of q_2 between systems (21*a*), (21*b*) and (21*b*), (21*c*) leads respectively to one equation $E_3(q_1, E) = 0$ of order three depending on the energy E, and one equation $E_4(q_1) = 0$ of order four independent of E. Then the linear combination $E_3 - 2E_4$ factorizes out $q_{1,\xi\xi}$, and the quotient is particularly simple

$$q_{1,\xi\xi\xi\xi} + (8\alpha - 2\beta) q_1 q_{1,\xi\xi} - 2(\alpha + \beta) q_{1,\xi}^2 - \tfrac{20}{3}\alpha\beta q_1^3)$$
$$+ (c_1 + 4c_2) q_{1,\xi\xi} + (6\alpha c_1 - 4\beta c_2) q_1^2 + 4c_1 c_2 q_1 + 4\alpha E = 0. \qquad (22)$$

Although this ODE, first obtained in the particular case $c_1 = c_2 = 0$ [22], contains some extraneous solutions as compared with those of the HH system, it does not contain 'too much' of them, in the sense that no extraneous polynomial degree for q_1 solution of (22) will be found. Moreover, the extraneous solutions q_1 of polynomial type are quite easy to filter out: indeed, in such a case the HH system implies not only that q_2^2 is polynomial, as seen from $(21b)$, but also that q_2 is polynomial, for the polynomial factorization of q_2^2 can only contain squares, due to the Hamiltonian invariant $(21a)$. A necessary filter, to be applied after obtaining solutions for (22), is then the condition that the square root of $q_{1,\xi\xi} + c_1 q_1 - \beta q_1^2$ be a polynomial.

Let us make a last important remark before undertaking the computation. The ODE (22) can also be viewed as the stationary reduction $(x, t) \to x$ of a conservative PDE

$$v_t + (v_{xxxx} + (8\alpha - 2\beta) v v_{xx} - 2(\alpha + \beta) v_x^2 - \tfrac{20}{3}\alpha\beta v^3)_x = 0 \qquad (23)$$

written here in the particular case $c_1 = c_2 = 0$, for which important results are already known. The only values of $\frac{\beta}{\alpha}$ for which equation (23) has an infinite number of conservation laws are [23] $(-1, -6, -16)$, corresponding respectively to the Sawada–Kotera [24] (SK), higher-order Korteweg–de Vries [25] (KdV5) and Kaup–Kupershmidt [13, 14] (KK) equations. Considering the more general reduction $(x, t) \to x - ct$ of PDE (23) only adds a term $-cq_1$ to the LHS of (22) and allows to check some already known solitary wave solutions of (23).

To summarize this introduction, let us handle at the same time the HH system $(21abc)$ and the solitary waves of PDE (23), by considering the single fourth order ODE (22) including the additional term $-cq_1$ on its LHS.

At the first step, if q_1 has a global degree P in (σ, τ), then $q_{1,\xi\xi\xi\xi}$ has degree $P + 4$, $q_1 q_{1,\xi\xi}$ and $q_{1,\xi}^2$ have degree $2P + 2$, and q_1^3 has degree $3P$; the only possibility for P is $P = 2$.

In the second step, we look for solutions q_1 under the form (15), restricted for simplicity to $\theta' = \text{constant} = k$ and constant coefficients $c_{j,l}$; for simplicity again, let us restrict q_1 to $q_1 = u_x$, where u, which is physically the field of the potential form of the conservative equation (23), is defined as

$$u = k(c_{00}\theta + c_{10}\sigma(\theta) + c_{01}\tau(\theta)) \qquad \theta = k(x - ct). \tag{24}$$

One then builds the twelve equations for the (constant) coefficients c_{00}, c_{10}, c_{01} and parameters μ_0, c, k. Equation $E_{0,0} = 0$ provides the value for the energy

$$E = \frac{1}{4\alpha}(c - 4c_1c_2)k^2c_{00} - \frac{1}{2\alpha}(3\alpha c_1 - 2\beta c_2)k^4c_{00}^2 + \frac{5}{3}\beta k^6 c_{00}^3. \tag{25}$$

The highest degree determining equations are

$$E_{5,1} \equiv 20c_{1,0}(\tfrac{1}{3}\alpha\beta c_{1,0}^2 - 6 + 2(2\alpha - \beta)c_{0,1} + \alpha\beta c_{0,1}^2) = 0 \tag{26}$$

$$E_{6,0} \equiv 20((2\alpha - \beta)(c_{1,0}^2 + c_{0,1}^2) + \alpha\beta c_{0,1}(c_{1,0}^2 + \tfrac{1}{3}c_{0,1}^2) - 6c_{0,1}) = 0 \tag{27}$$

and they yield five possible leading behaviours

$$\begin{aligned} &(c_{1,0}, c_{0,1}) = (0, 3/\alpha) \qquad (0, -6/\beta) \qquad (\pm 3/2\alpha, 3/2\alpha) \\ &(\mp 3/\beta, -3/\beta) \qquad (\pm(3/2\alpha + 3/\beta), 3/2\alpha - 3/\beta). \end{aligned} \tag{28}$$

We discard the third and fourth leading behaviours, for u would only depend on $\tau \mp \sigma$.

At the third step, $\mu_0(\mu_0^2 - 1) \neq 0$, one finds two solutions which we label with an m to indicate their dependence on 'modified' hyperbolic functions, i.e. μ_0 arbitrary

$$(\text{SK}_m): \qquad \frac{\beta}{\alpha} = -1 \qquad c_2 = c_1$$
$$u = \frac{k}{\alpha}\left[3\tau - \frac{1 + c_1k^{-2}}{2}\theta\right] \qquad c = k^4 - c_1^2 \tag{29a}$$

$$(\text{KK}_m): \qquad \frac{\beta}{\alpha} = -16 \qquad c_2 = \frac{c_1}{16}$$
$$u = \frac{k}{8\alpha}\left[3\tau - (1 - \frac{3}{4}\mu_0^2 + \frac{1}{4}c_1k^{-2})\theta\right] \tag{29b}$$
$$c = \left[1 - \frac{15}{4}\mu_0^2\left(1 - \frac{3}{4}\mu_0^2\right)\right]k^4 - \frac{c_1^2}{16}.$$

At the fourth step, $\mu_0 = 0$, one finds again the two above solutions in the particular case $\mu_0 = 0$, plus two solutions for arbitrary $\frac{\beta}{\alpha}$

$$(\mathrm{HH}_1): \quad \frac{\beta}{\alpha} \neq -1 \qquad \mu_0 = 0$$
$$u = \frac{k}{\alpha}\left[3\tau - \left(1 - \frac{c_1\alpha - (2\alpha+\beta)c_2}{4(\alpha+\beta)}k^{-2}\right)\theta\right]$$
$$c = -\left(1 + \frac{3\beta}{4\alpha}\right)(16k^4 - c_1^2) + \frac{(3\beta - 2\alpha)(\beta + 2\alpha)(c_2 - c_1)[(2\alpha+\beta)c_1 + \beta c_2]}{4\alpha(\alpha+\beta)^2} \tag{30a}$$

$$(\mathrm{HH}_2): \quad \mu_0 = 0 \qquad u = -\frac{k}{\beta}\left[6\tau - 2(1 + \tfrac{1}{4}c_1 k^{-2})\theta\right]$$
$$c = -\frac{\alpha}{\beta}(16k^4 - c_1^2) \tag{30b}$$

plus three solutions which exist only in integrable cases (SK, KdV5)

$$(\mathrm{SK}_1): \quad \frac{\beta}{\alpha} = -1 \qquad c_2 = c_1 \qquad \mu_0 = 0$$
$$c_{0,0}\ \text{arbitrary} \qquad u = \frac{3k}{\alpha}\tau + kc_{0,0}\theta$$
$$c = 4(k^2 + c_1)(4k^2 + c_1) + 20\alpha c_{0,0}k^2[(2 + \alpha c_{0,0})k^2 + c_1] \tag{31a}$$

$$(\mathrm{KdV5}_1): \quad \frac{\beta}{\alpha} = -6 \qquad \mu_0 = 0 \qquad c_{0,0}\ \text{arbitrary}$$
$$u = \frac{k}{\alpha}\tau + kc_{0,0}\theta$$
$$c = 4(k^2 + c_2)(4k^2 + c_1) + 4\alpha c_{0,0}k^2[20k^2 + 30\alpha c_{0,0}k^2 + 3(c_1 + 4c_2)] \tag{31b}$$

$$(\mathrm{KdV5}_2): \quad \frac{\beta}{\alpha} = -6 \qquad \mu_0 = 0$$
$$u = \frac{k}{\alpha}\left[2\tau \pm \sigma - \left(\frac{1}{2} + \frac{c_1 + 4c_2}{20k^2}\right)\theta\right] \tag{31c}$$
$$c = 21k^4 - \tfrac{1}{10}(3c_1^2 - 16c_1c_2 + 48c_2^2)$$

plus the confluence, which occurs for $\frac{\beta}{\alpha} = -2$, of the two solutions HH_1 and HH_2.

This completes the list of solutions of ODE (22) (plus its term $-cq_1$) obtainable by our method from the restricting assumption (24); in all of them, k is arbitrary.

The method of summing a Taylor series would miss the following solutions: SK_m and KK_m because of the difficulty of summing the series, HH_1, HH_2 and $\mathrm{KdV5}_2$ because c and k are not linked by the dispersion relation.

The method of a Riccati subequation would only find solutions polynomial in tanh, i.e. would miss solutions SK_m, KK_m and KdV5_2.

The method of an elliptic subequation would miss the same three solutions; with a degenerate elliptic subequation [8], it would only find pure sech or pure tanh solutions; but, with a non-degenerate elliptic subequation [9], it would find in addition the solution obtained from (31*a*) by replacing τ by $\frac{k}{3}+\frac{1}{k}\zeta(\frac{\theta}{k},\frac{4}{3}k^4,g_3)$, g_3 arbitrary, where ζ is the Weierstrass elliptic function.

In the particular case $c_1=c_2=0$, those of the above solutions which in addition satisfy the condition $u_x\to 0$ for $\theta\to\pm\infty$, i.e. $c_{0,0}=0$, are the one-soliton solutions of the three integrable cases, namely (29*b*) for $\mu_0^2=\frac{4}{3}$, (31*a*) and (31*b*) for $c_{0,0}=0$.

Exact solutions to the Hénon–Heiles Hamiltonian system are found by imposing on the above list (29)–(31) the two additional conditions: $c=0$ and $q_{1,\xi\xi}-c_1q_1-\beta q_1^2$ the square of a polynomial in (σ,τ); we have checked that these two conditions are sufficient to get rid of all extraneous solutions. As to the exact solutions to the coupled PDE system (18*ab*), they are given by imposing these two conditions, plus the two conditions $q_1\to 0$ (i.e. $c_{0,0}=0$) and $q_2\to 0$ when $\xi\to\pm\infty$.

Table 1 gathers the solutions obtained in this way. Solution 1 is the one-soliton solution to the pure Boussinesq equation. Solutions 2, 4, 7 to (18) were already found by Hase and Satsuma [18], and later rediscovered by Rao and Kaup [20], who mentioned the link to the HH system. Solution 6 (μ_0 arbitrary) is to our knowledge a new solution to the coupled system (18).

These results are an indication for the possible existence of an additional first integral of the HH system for the values of $\frac{\beta}{\alpha}$ and $\frac{c_2}{c_1}$ numbered 2 to 5 in table 1.

4.2. *Zhiber–Shabat equation*

The equation

$$U_{xt}+\alpha e^U+a_1e^{-U}+a_0e^{-2U}=0 \qquad \alpha\neq 0 \tag{32}$$

includes as particular cases Liouville ($a_1=a_0=0$), sine- (or rather sinh)-Gordon (SG: $a_1\neq 0, a_0=0$) or Dodd–Bullough–Mikhailov [26] (DBM: $a_1=0, a_0\neq 0$) equations. It is polynomial in the variable $u=e^U$

$$uu_{xt}-u_xu_t+\alpha u^3+a_1u+a_0=0. \tag{33}$$

Its reduction $(x,t)\to\xi$ can be integrated once

$$-\tfrac{1}{2}cu_\xi^2+\alpha u^3-3C_1u^2-a_1u-\tfrac{1}{2}a_0=0 \qquad C_1 \text{ arbitrary} \tag{34}$$

and possesses the general two-parameter solution

$$u=\frac{C_1}{\alpha}+\frac{2c}{\alpha}\wp\left(\xi-\xi_0,\frac{9C_1^3+\alpha a_1}{c^2},\frac{4C_1^3+2\alpha a_1C_1+\alpha^2a_0}{4c^3}\right) \tag{35}$$

linear in the Weierstrass elliptic function $\wp$.

If we look for solutions of (33) with our method, we can only find one-parameter solutions, corresponding to the degeneracy of $\wp$ into a hyperbolic function. Let us check it as an example. With the simplifying assumption $\theta'=k$ and $c_{j,l}$ constant, we take

$$u=k^2(c_{2,0}\sigma^2+c_{1,1}\sigma\tau)+k(c_{1,0}\sigma+c_{0,1}\tau)+c_{0,0} \tag{36}$$

Table 1. Each line defines a solution to the HH system (21), with indication in column 2 of the 'mother solution' as labelled in formulae (29)–(31); the value of N is $N = 9\epsilon^{-1}\alpha^{-3}(\beta+2\alpha)k^4$. Column '$c = c_{00} = 0$ iff' contains the condition for this line to also define a solitary wave solution to the Bq–NLS system (18). Line 6 defines a new solitary wave solution to the Bq–NLS system in terms of the 'modified' hyperbolic functions (μ_0 arbitrary). In line 8, k^2 must be negative ($k^2 = -\kappa^2 < 0$) for q_1 and q_2 to be real: indeed, the choice $\theta = k\xi + \mathrm{i}\frac{\pi}{2}$ yields $\sigma = 1/\sin(\kappa\xi)$, $\tau = \cot\kappa\xi$.

	Hénon–Heiles: $c = 0,\ q_1 = k^2\ (c_{01}\tau' + c_{10}\sigma' + c_{00})$									Boussinesq–NLS: $c = 0, c_{00} = 0$		
#	Mother solution	$\frac{\beta}{\alpha}$	$\frac{c_2}{c_1}$	μ_0	k^2	αc_{01}	αc_{10}	αc_{00}	q_2^2	$c = c_{00} = 0$ iff	αq_1	q_2^2
1	HH_2	arb	arb	0	$\pm\frac{c_1}{4}$	$-6\frac{\alpha}{\beta}$	0	$2\frac{\alpha}{\beta}\left[1+\frac{c_1}{4k^2}\right]$	0	$4k^2 = -c_1$	$6\frac{\alpha}{\beta}k^2\sigma^2$	0
2	HH_1	arb	1	0	$\pm\frac{c_1}{4}$	3	0	$-\left[1+\frac{c_1}{4k^2}\right]$	$N\left[\sigma^2+\frac{4k^2+c_1}{12k^2}\right]^2$	$4k^2 = -c_1$	$-3k^2\sigma^2$	$N\sigma^4$
3	HH_1	arb	$-\frac{\beta+2\alpha}{\beta}$	0	$\pm\frac{c_1}{4}$	3	0	$-\frac{1}{16}\left[1-\frac{(\beta+4\alpha)c_1}{4\beta k^2}\right]$	$N\left[\sigma^2+\frac{4k^2-c_1}{12k^2}\right]^2$	$\frac{\beta}{\alpha} = -2,\ 4k^2 = -c_1$	$-3k^2\sigma^2$	0
4	HH_1	arb	$-\frac{\alpha}{2\alpha+3\beta}$	0	$-c_2$	3	0	0	$N(\sigma\tau)^2$		$-3k^2\sigma^2$	$N(\sigma\tau)^2$
5	HH_1	arb	$-\frac{\alpha(5\beta+4\alpha)}{\beta(3\beta+2\alpha)}$	0	$-\frac{\alpha c_1}{2\alpha+3\beta}$	3	0	$-\frac{3\beta+2\alpha}{\beta}$	$N(\sigma\tau)^2$	$\frac{\beta}{\alpha} = -\frac{2}{3}$	$-3k^2\sigma^2$	$N(\sigma\tau)^2$
6	SK_m	-1	1	arb	$\pm c_1$	3	0	$-\frac{1}{2}\left[1+\frac{c_1}{k^2}\right]$	$N(\sigma\tau)^2$	$k^2 = -c_1$	$3k^2(-\sigma^2+\mu_0\sigma)$	$N(\sigma\tau)^2$
7	$KdV5_1$	-6	arb	0	$-c_2$	1	0	0	$\frac{c_1-4c_2}{36c_2}N\sigma^2$		$-k^2\sigma^2$	$\frac{c_1-4c_2}{36c_2}N\sigma^2$
8	$KdV5_2$	-6	$\frac{-46\pm2\sqrt{253}}{9}$	0	$\frac{6c_2-c_1}{10}$	2	±1	$\frac{c_2}{c_1-6c_2}$	$\frac{N}{2}\left[\sigma^2\mp\sigma\tau+\frac{1}{2}\right]^2$	impossible	—	—

and find two leading behaviours

$$(c_{1,1}, c_{2,0}) = (0, 2c/\alpha) \qquad (\pm c/\alpha, c/\alpha). \tag{37}$$

The first one yields the solution

$$u = \frac{2}{\alpha} ck^2\sigma^2 + c_{0,0} \qquad \mu_0 = 0 \tag{38a}$$

$$\alpha c_{0,0}^3 + a_1 c_{0,0} + a_0 = 0 \tag{38b}$$

$$4ck^2 c_{0,0} - 3\alpha c_{0,0}^2 - a_1 = 0 \tag{38c}$$

while the second one leads to

$$u = \frac{1}{2\alpha} ck^2 \left[(\tau \pm \sigma)^2 - 1\right] + c_{0,0} \qquad \mu_0 \text{ arbitrary} \tag{39a}$$

$$\alpha c_{0,0}^3 + a_1 c_{0,0} + a_0 = 0 \tag{39b}$$

$$ck^2 c_{0,0} - 3\alpha c_{0,0}^2 - a_1 = 0 \tag{39c}$$

i.e. a solution identical to the first one.

In the linearizable Liouville case $a_1 = a_0 = 0$, coefficient $c_{0,0}$ is zero and (c, k) arbitrary. In all other cases, coefficient $c_{0,0}$ is non-zero and characterizes the vacuum state $U_0 = \log c_{0,0}$; a direct linearization of (32) about $U = U_0$ provides the dispersion relation

$$4ck^2 = \alpha c_{0,0} - a_1 c_{0,0}^{-1} - 2a_0 c_{0,0}^{-2} \tag{40}$$

and the determining equation (38*c*) is simply a linear combination of the vacuum equation (38*b*) and the dispersion relation (40).

The solitary wave $U - U_0$ satisfies the boundary condition $U - U_0 \to 0$ when $\xi \to \pm\infty$. In the two particular cases SG and DBM, one gets the one-soliton

$$\text{SG:} \quad a_0 = 0 \qquad a_1 \neq 0 \qquad c_{0,0}^2 = -\frac{a_1}{\alpha} \qquad ck^2 = \frac{\alpha}{2} c_{0,0}$$
$$u = c_{0,0} \tanh^2(\theta) \tag{41}$$

$$\text{DBM:} \quad a_0 \neq 0 \qquad a_1 = 0 \qquad c_{0,0}^3 = -\frac{a_0}{\alpha} \qquad ck^2 = \frac{3\alpha}{4} c_{0,0}$$
$$u = \frac{1}{2} c_{0,0} \left[3\tanh^2(\theta) - 1\right]. \tag{42}$$

4.3. *Dispersive equation with two monomial nonlinearities*

The generalized Korteweg–de Vries equation

$$U_t + (\alpha + \beta U^\gamma) U^\gamma U_x + \delta U_{xxx} = 0 \qquad \alpha\beta\gamma\delta \neq 0 \tag{43}$$

has been encountered in plasma physics, wave phenomena and astrophysics for $\gamma = 1$ (Zabusky [27]), $\gamma = \frac{1}{2}$ (Schamel [28]), $\gamma = 2$ (Chandrasekhar [29]). The reduction

$(x, t) \to \xi = x - ct$ can be integrated twice to yield, after the change of function $u = U^\gamma$ and with the restriction $\gamma \neq -1, -2, -\frac{1}{2}$,

$$\frac{\delta}{2\gamma^2} u_\xi^2 - \frac{c}{2} u^2 + \frac{\alpha}{(\gamma+1)(\gamma+2)} u^3 + \frac{\beta}{(2\gamma+1)(2\gamma+2)} u^4 - C_1 u^{2-(1/\gamma)} - C_2 u^{2-(2/\gamma)} = 0. \tag{44}$$

Due to the ξ translational invariance, any particular non-constant solution depends on one arbitrary constant ξ_0 and represents the general solution.

According to a classical result (see, e.g., Hille [30, ch 11]), the general solution of (44) is single valued iff the equation is of Briot-Bouquet type, i.e. u_ξ^2 equal to a polynomial of degree three or four in u. This occurs only for values of (γ, C_1, C_2) equal to $(1, \text{arbitrary}, \text{arbitrary})$, $(\frac{1}{2}, \text{arbitrary}, 0)$, $(2, 0, \text{arbitrary})$, $(\text{arbitrary}, 0, 0)$, i.e. precisely the cases of physical interest, plus the obvious case $C_1 = C_2 = 0$ (Hereman and Takaoka [5]). In all four cases, which can evidently be treated as just one case, the general solution of (44) is elliptic, degenerate or not (see e.g. Wadati [31] in the case $\gamma = 1$). The first case is the only one allowing both C_1 and C_2 to be arbitrary, and this proves that the reduction $(x, t) \to \xi$ of the modified Korteweg–de Vries equation has a single valued general solution.

Particular solutions polynomial in (σ, τ) are single valued and therefore exist only when (44) is of Briot-Bouquet type. They have degree $P = 1$ and they are found *without computation*, just by identifying (44) and (11); they exist only when the polynomial of degree four has a multiple zero, in which case they are either linear in σ for one double and two simple zeros, or proportional to $(1 \pm \tanh)$ for two double zeros.

For instance, for $\gamma = \frac{1}{2}$, one finds three solutions: the one of Tagare and Chakrabarty [32]

$$\begin{aligned} &\gamma = \frac{1}{2} \qquad C_1 = 0 \qquad C_2 = 0: \; u = \left[-\frac{3c}{\beta}\right]^{1/2} \sigma \\ &c \text{ arbitrary} \qquad k^2 = \frac{c^3}{\delta} \qquad \mu_0^2 = -\frac{16\alpha^2}{75\beta c} \end{aligned} \tag{45}$$

a second one depending on one arbitrary parameter λ

$$\begin{aligned} &\gamma = \frac{1}{2} \qquad C_1 = -\frac{1}{3750}(\lambda-3)^3(\lambda+1) \\ &C_2 = 0: \; u = \frac{\alpha}{5\beta}[\lambda - 3 \pm 2\sqrt{\lambda(\lambda-3)}\sigma] \\ &k^2 = -\frac{2^4\alpha^6}{3^3 5^6 \beta^3 \delta}\lambda(\lambda-3)^3(\lambda+3)^2 \qquad c = \frac{2\alpha^2}{75\beta}(\lambda^2-9) \\ &\mu_0^2 = \frac{(\lambda-1)^2}{\lambda(\lambda-3)} \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\gamma = \frac{1}{2} \qquad C_1 = 0 \qquad C_2 = 0: u = -\frac{2\alpha}{5\beta}[1 \pm \tanh(\theta)] \\ &\mu_0 = 0 \qquad c = -\frac{16\alpha^2}{75\beta} \qquad k^2 = -\frac{2^{10}\alpha^6}{3^3 5^6 \beta^3 \delta}. \end{aligned} \tag{47}$$

The third one is the common value of the first and second ones for $\mu_0^2 = 1$, i.e. $\lambda = -1$.

The method of summation of exponentials has provided some [5, 33] of these solutions after lengthy computations, but it has failed to provide any solution with $\mu_0(\mu_0^2 - 1) \neq 0$.

4.4. *Dispersive equation with one monomial nonlinearity*

Let us set $\alpha = 0$ in the previous example, (43). Then, the twice integrated form (44) remains valid, with $\gamma \neq -1, -\frac{1}{2}$. The requirement that (44) with $\alpha = 0$ be of Briot-Bouquet type provides, in addition to the four previous cases, $(\gamma, C_1, C_2) = (-2, 0, \text{arbitrary})$ which corresponds to the Ermakov [34] or Pinney [35] equation, which is linearizable, see next example. The invariance of (44) under $u \to -u$ for $\alpha = 0, C_1 = 0, C_2 = 0$ suggests considering the transformed equation in $v = u^2 = U^{2\gamma}$

$$\frac{\delta}{8\gamma^2} v_\xi^2 - \frac{c}{2} v^2 + \frac{\beta}{(2\gamma+1)(2\gamma+2)} v^3 - C_1 v^{2-(1/2\gamma)} - C_2 v^{2-(1/\gamma)} = 0. \qquad (48)$$

This equation is of Briot-Bouquet type only for five values of (γ, C_1, C_2), equal to: $(\frac{1}{2}, \text{arbitrary}; \text{arbitrary})$; $(\frac{1}{4}, \text{arbitrary}, 0)$; $(-\frac{1}{4}, \text{arbitrary}, 0)$; $(1, 0, \text{arbitrary})$; and $(\text{arbitrary}, 0, 0)$.

The first case is the only one allowing both C_1 and C_2 to be arbitrary, and this proves that the reduction $(x, t) \to \xi$ of the Korteweg–de Vries [36] equation has a single valued general solution.

In the second case, v is a Weierstrass elliptic function. In the third case, v is a degenerate Jacobi elliptic function proportional to either $\operatorname{sechm} \theta$ or $(1 \pm \tanh \theta)$, depending on C_1. As to the last two cases, they have already been found by considering the form (44).

4.5. *Ermakov–Pinney equation*

After the reduction $(x, t) \to \xi$, one integration and the setting of C_1 to zero, (43) reads for $\alpha = 0, \gamma = -2$

$$\delta U_{\xi\xi} - cU - \tfrac{1}{3}\beta U^{-3} = 0 \qquad (49)$$

which we rewrite for convenience as

$$U_{\xi\xi} - a^2 U + b^2 U^{-3} = 0. \qquad (50)$$

This defines the Ermakov [34] or Pinney [35] equation. Its first integral (44), written in the variable $u = U^{-2}$

$$\tfrac{1}{8} u_\xi^2 - \tfrac{1}{2} a^2 u^2 - C_2 u^3 - \tfrac{1}{2} b^2 u^4 = 0 \qquad (51)$$

can be identified to equation (11), linearizable into (13); this defines the general solution of Ermakov–Pinney equation as

$$U = \left[A + B\mathrm{e}^{2a\xi} + C\mathrm{e}^{-2a\xi}\right]^{1/2}$$
$$4BC - A^2 = -b^2/a^2 \qquad (52)$$

where A, B and C are integration constants.

Many proofs of this result have been given, among them two straightforward proofs *via* Painlevé analysis [37]. Let us use our method to give another proof of the linearizability of Ermakov–Pinney equation, by considering this equation written in the variable $u = U^{-2}$

$$-\tfrac{1}{2}uu_{\xi\xi} + \tfrac{3}{4}u_\xi^2 - a^2u^2 + b^2u^4 = 0. \tag{53}$$

Looking for

$$u = c_{1,0}(\xi)\sigma(\theta(\xi)) + c_{0,1}(\xi)\tau(\theta(\xi)) + c_{0,0}(\xi) \qquad \theta' \neq 0 \tag{54}$$

we find three leading behaviours

$$\begin{aligned}&(c_{1,0}, c_{0,1}) = (0, \varepsilon\theta'/2b) \qquad (\varepsilon\theta'/2b, 0) \qquad (\varepsilon\theta'/4b, \varepsilon'\theta'/4b)\\ &\varepsilon^2 = \varepsilon'^2 = 1\end{aligned} \tag{55}$$

of which the third one must be discarded.

Let us only solve the first case. At the third step, i.e. $\mu_0(\mu_0^2 - 1) \neq 0$, one finds nothing. At the fourth step, i.e. $\mu_0 = 0$, one finds the single solution

$$u = (\theta'/2b)\tanh(\theta) + c_{0,0} \qquad \mu_0 = 0 \tag{56}$$

in which $b^{-1}\theta'$ satisfies (53), and $c_{0,0}$ the Riccati equation

$$c_{0,0}' - 2bc_{0,0}^2 - (\theta''/\theta')c_{0,0} + (1/2b)\theta'^2 = 0. \tag{57}$$

Taking for θ the particular solution $\theta = a(\xi - \xi_1)$, we obtain

$$U^{-2} = u = (a/2b)\left[\tanh(a(\xi - \xi_1)) - \tanh(a(\xi - \xi_2))\right] \tag{58}$$

an expression which depends on two arbitrary constants ξ_1, ξ_2 and is the general solution of (53). It is also equal to

$$u = \frac{W}{2b\psi_1\psi_2} \tag{59}$$

where ψ_1 and ψ_2 are two linearly independent solutions of $\psi_{\xi\xi} - a^2\psi = 0$, and W their constant Wronskian.

Other examples can be found in Musette and Conte [38], in particular solitary wave solutions associated with the nonlinear Schrödinger equation and the Boussinesq equation.

5. Conclusion

The introduction of a projective Riccati system as subequations of a nonlinear ODE of order greater than one provides particular solutions by the determination of a *finite* number of coefficients. This prevents the drawback of having to sum entire series in exponential solutions of the linearized equation. With the simplifying assumption of constant coefficients, one finds as solutions polynomials in two elementary bell-shaped and kink-shaped functions; this covers the large majority of physically interesting solitary waves. Without this simplifying assumption, one finds more solutions, and one can even find the general solution of some ODEs.

Physics sometimes provides systems of differential equations which cannot be converted to polynomial form, or for which one is unable to find polynomial forms yielding an integer value for the global degree P in (σ, τ). In such a case, which reflects multivaluedness intrinsic to the equation, our method, based on single valuedness assumptions, is of no help. One could of course devise some asymptotic expansion, but this would bring us back to situations where an *infinite* set of coefficients must be determined. Such an interesting system, where no solution is known in closed form although numerical evidence indicates a physically acceptable solution, is provided by a Langmuir plasma [39].

The present method can evidently be generalized to any subequation, which must be defined in its canonical reduced form [7], e.g. the Riccati or elliptic equations.

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